WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 3 - February 10th, 2016
Round 1: Graphing on a Number Line

All answers must be in simplest exact form in the answer section

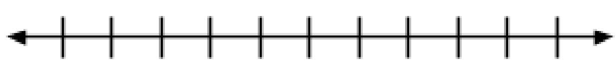
NO CALCULATOR ALLOWED

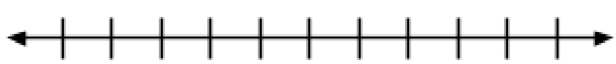
1. Graph: $|x| - 2 < 4$

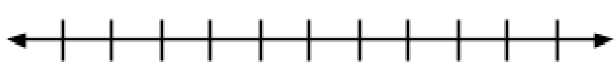
2. Graph: $2x - 5 \leq 3x + 0.5 < x$

3. Graph: $2x^2 - 4x + 6 > (x+4)(x-1)$

ANSWERS

(1 pt.) 1. 

(2 pts.) 2. 

(3 pts.) 3. 

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 3 - February 10th, 2016
Round 2: Operations on Polynomials

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Expand and simplify: $(3x-5)(x+9)$

2. If $(x-3y)^4 = Ax^4 + Bx^3y + Cx^2y^2 + Dxy^3 + Ey^4$, determine the value of D.

3. Factor: $4x^4 - x^2 - 4x^2y^2 + y^2$

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

Freshman Meet 3 - February 10th, 2016
Round 3: Techniques of Counting and Probability

All answers must be in simplest exact form in the answer section
NO CALCULATOR ALLOWED

1. Five separate points lie on the circumference of a circle. How many different triangles can be transcribed in the circle by joining three of these points at a time?

2. A bag contains 4 blue pens, 6 black pens, and 8 red pens. Three pens are chosen without replacement. What is the probability that the three pens chosen will all be different colors?

3. There is a bag with 4 white marbles and 1 gold marble. Erica and Jim take turns drawing one marble without replacement until the bag is empty. If Jim draws first, what is the probability that Erica draws the gold marble?

ANSWERS

(1 pt.) 1. ____________________ possible triangles

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

WORCESTER COUNTY MATHEMATICS LEAGUE

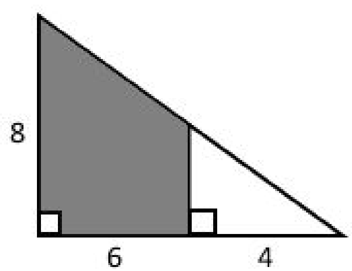
Freshman Meet 3 - February 10th, 2016
Round 4: Perimeter, Area, and Volume

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. A rectangle and a square have the same perimeter. If the width of the rectangle is half the length of the square's side and the area of the square is 256 square units, what is the area of the rectangle?

2. Find the area of the shaded region in the following diagram.



3. In a right rectangular prism, the length, width, and height are in the ratio 4:2:1. The volume of the prism is 512 cubic units. What is the surface area of the prism?

ANSWERS

(1 pt.) 1. ____________________________ sq. units

(2 pts.) 2. ___________________________ sq. units

(3 pts.) 3. ____________________________ sq. units

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 3 - February 10th, 2016

Team Round

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (3 points each)

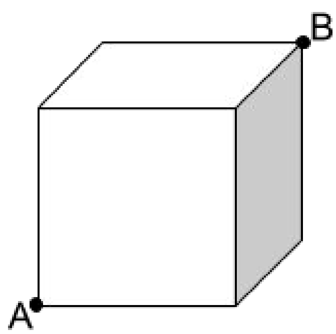
APPROVED CALCULATORS ALLOWED

1. On the real number line, let $A = \{x: \ -4 < 3 - 5x < 18\}$ and $B = \{x: \ |x-3| > 5\}$. If A^C denotes the complement of set A, graph the following region: $(A \cup B) \cap (A \cap B)^C$.

2. Factor completely: $64x^6 - 1$

3. A jar contains 5 quarters and 3 dimes. If Isaac shakes out two coins at random, what is the probability that the total value of the two coins is 35 cents?

4. In the following cube, if the straight line distance between A and B is $8\sqrt{3}$ units, what is the cube's surface area?



5. How many integers strictly larger than 999 and strictly less than 10,000 do not contain the digit 4?

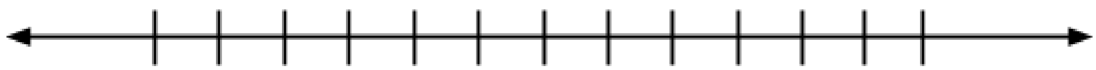
6. If $9a^3 + 5a - x$ is divided by $3a - 2$, the remainder is -2. Determine x.

7. A sequence of 14 single digit numbers has the property that the sum of any three consecutive numbers is equal to 20. If the fourth number is 9 and the twelfth number is 7, what is the product of the first four digits?

8. Jack spins a fair spinner three times and draws a dot at the point of the circle where the tip of the spinner stops each time. After he is done spinning, what is the probability that he can find a semicircle on the circle which contains all three points?

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 3 - February 10th, 2016
Team Round Answer Sheet

1. 

2. ____________________________

3. ____________________________

4. ____________________________ square units

5. ____________________________

6. ____________________________

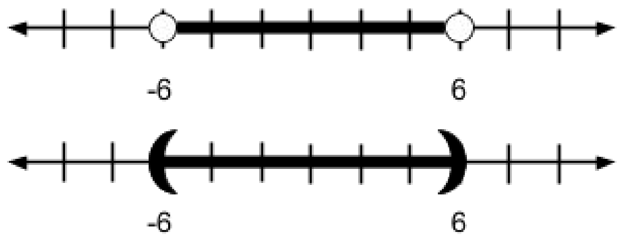
7. ____________________________

8. ____________________________

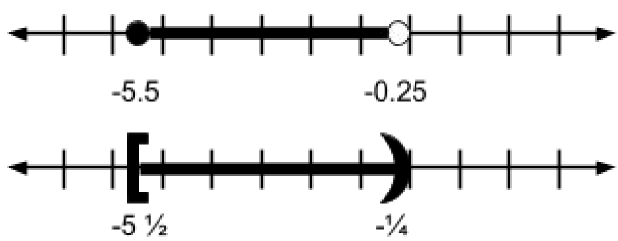
WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 3 - February 10th, 2016 ANSWER KEY

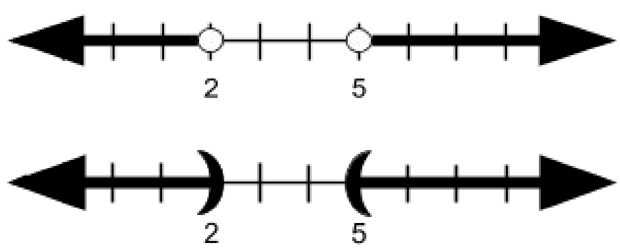
Round 1:

1. 

(Auburn)

2. 

(Quaboag)

3. 

(Douglas)

Round 2:

1. $3x^2 + 22x - 45$	(St. Peter-Marian)
2. -108	(Westboro)
3. $(x+y)(x-y)(2x+1)(2x-1)$	(Shepherd Hill)

Round 3:

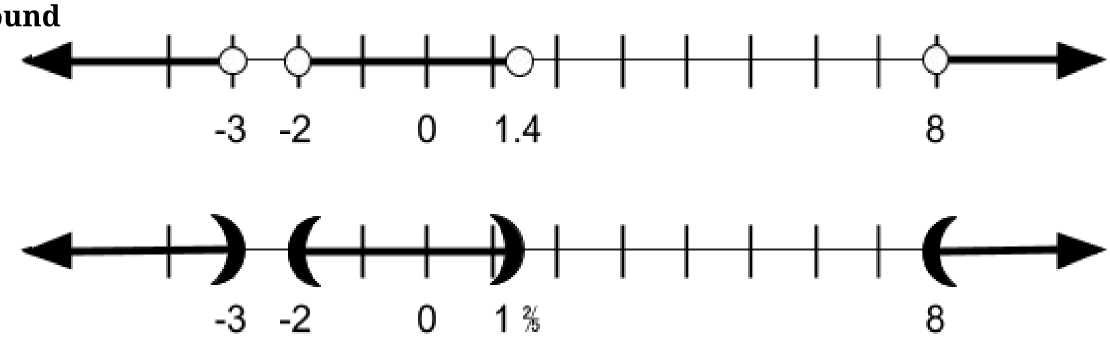
1. 10	(Nashoba)
2. $\frac{4}{17}$	(Douglas)
3. $\frac{2}{5}$ or 0.4	(South)

Round 4:

1. 192 (Assabet Valley)
2. $33\frac{3}{5}$ or 33.6 (Shrewsbury)
3. 448 (Southbridge)

TEAM Round

1.



(West Boylston)

2. $(2x+1)(2x-1)(4x^2+2x+1)(4x^2-2x+1)$ (St. Peter-Marian)

3. $\frac{15}{28}$ or 0.536 (Assabet Valley)

4. 384 (Shrewsbury)
5. 5832 (Quaboag)
6. 8 (Bartlett)
7. 2268 (Hudson)

8. $\frac{3}{4}$ or 0.75 (AMSA)

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 3 - February 10th, 2016 - SOLUTIONS
Round 1: Graphing on a Number Line

1. Graph: $|x| - 2 < 4$

Solution: We have that

$$|x| - 2 < 4$$
$$|x| < 6$$

Therefore we know that $x < 6$ or $x > -6$, which we can rewrite as

$$-6 < x < 6.$$

2. Graph: $2x - 5 \leq 3x + 0.5 < x$

Solution: Begin by splitting the joint inequality into two separate inequalities and then solve each inequality on its own.

$2x - 5 \leq 3x + 0.5$	AND	$3x + 0.5 < x$
$-5 \leq x + 0.5$	AND	$2x + 0.5 < 0$
$-5.5 \leq x$	AND	$2x < -0.5$
$-5.5 \leq x$	AND	$x < -0.25$

We recombine the two inequalities to see that $-5.5 \leq x < -0.25$

3. Graph: $2x^2 - 4x + 6 > (x + 4)(x - 1)$

Solution: Begin by expanding and simplifying the right hand side of the inequality and then sum like terms:

$$2x^2 - 4x + 6 > (x + 4)(x - 1)$$
$$2x^2 - 4x + 6 > x^2 + 3x - 4$$
$$x^2 - 7x + 10 > 0$$

Now factor the left hand side to see that

$$(x - 5)(x - 2) > 0$$

For this inequality to hold, both factors must be strictly positive or both must be strictly negative. The only way that both factors would be strictly positive is if $x > 5$. Similarly, the only way both factors can be strictly negative is if $x < 2$. Therefore, the solution to the original inequality is $x > 5$ or $x < 2$.

Round 2: Operations on Polynomials

1. Expand and simplify: $(3x-5)(x+9)$

Solution: We expand the expression to see

$$(3x-5)(x+9)$$
$$3x^2-5x+27x-45$$
$$3x^2+22x-45.$$

2. If $(x-3y)^4 = Ax^4+Bx^3y+Cx^2y^2+Dxy^3+Ey^4$, determine the value of D.

Solution 1 (Direct Computation): We have that

$$\begin{aligned}(x-3y)^4 &= (x^2-6xy+9y^2)(x^2-6xy+9y^2)\\ &= x^4-6x^3y+9x^2y^2-6x^3y+36x^2y^2-54xy^3+9x^2y^2-54xy^3+81y^4\\ &= x^4-12x^3y+54x^2y^2-108xy^3+81y^4\end{aligned}$$

Solution 2 (Pascal's Triangle): To quickly determine the value of the coefficients in a binomial expansion, we can use Pascal's Triangle, which looks like

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Pascal's triangle shows that for a fourth order binomial expansion, such as $(x-3y)^4$, the coefficient on $x(-3y)^3$ would be 4. Therefore, we have that $D = 4\cdot(-3)^3 = -108.$

3. Factor: $4x^4-x^2-4x^2y^2+y^2$

Solution: Begin by pulling out a factor of x^2 from the first two terms and a factor of $-y^2$ from the second two terms to see that

$$4x^4-x^2-4x^2y^2+y^2$$
$$x^2(4x^2-1)-y^2(4x^2-1)$$
$$(x^2-y^2)(4x^2-1).$$

Now use the formula for a difference of squares to get

$$(x+y)(x-y)(2x+1)(2x-1).$$

Round 3: Techniques of Counting and Probability

1. Five separate points lie on the circumference of a circle. How many different triangles can be transcribed in the circle by joining three of these points at a time?

Solution: Regardless of which 3 points we pick on the circle we will always form a triangle. Moreover the order in which we pick the points is irrelevant. Therefore, to determine the total number of possible triangles we simply compute how many ways we can choose 3 points from a set of 5. This is given by 5 choose 3 or

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} =$$
$$\frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{(3\cdot 2\cdot 1)(2\cdot 1)} =$$
$$\frac{5\cdot 4}{2} = 10.$$

2. A bag contains 4 blue pens, 6 black pens, and 8 red pens. Three pens are chosen without replacement. What is the probability that the three pens chosen will all be different colors?

Solution: One way this event could occur is if we draw a red pen, then a black pen, and finally a blue pen. The probability of this particular draw occurring is given by

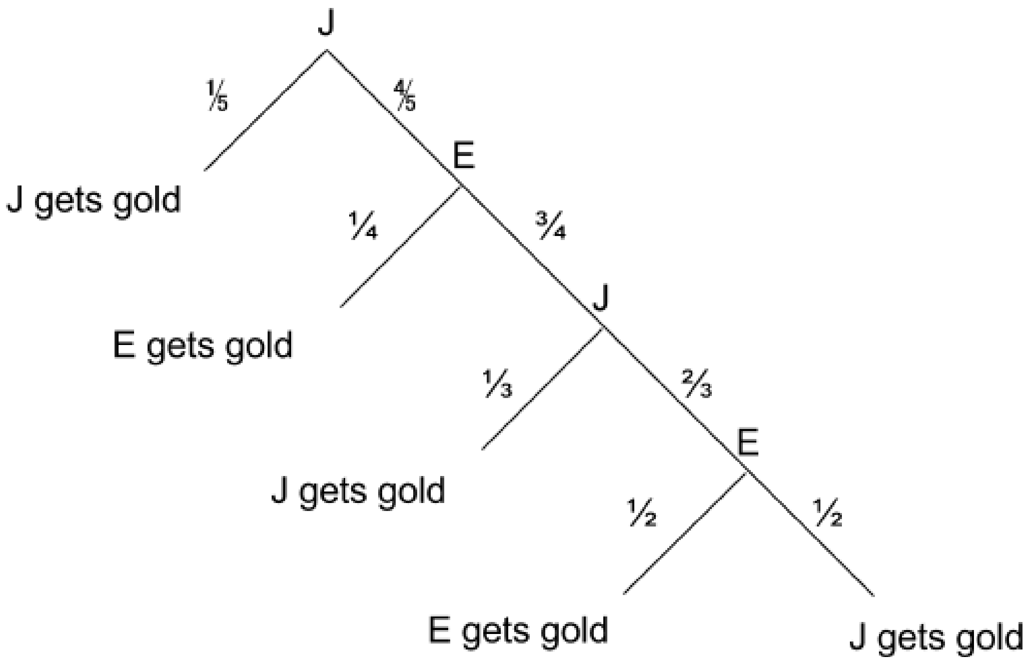
$$\frac{8}{18}\cdot\frac{6}{17}\cdot\frac{4}{16} =$$
$$\frac{1}{3}\cdot\frac{1}{17}\cdot\frac{2}{1} = \frac{2}{51}$$

There are precisely 3! or 6 possible ways we could pull out three pens with different colors, and each possibility has the same probability of occurring (which we computed above). Therefore, the total probability that we draw three different colored pens is

$$\frac{2}{51}\cdot 6 = \frac{12}{51} = \frac{4}{17}$$

3. There is a bag with 4 white marbles and 1 gold marble. Erica and Jim take turns drawing one marble without replacement until the bag is empty. If Jim draws first, what is the probability that Erica draws the gold marble?

Solution 1 (Probability Tree): We can picture this situation like a tree where each branch has a certain probability of occurring.



Turn 1 of the game is represented by the top of the tree. Jim (or J in the diagram) draws a marble and it has a 1/5 chance of being gold and a 4/5 chance of being white. Since we only effectively care about the gold marble, we can assume the game ends when the gold marble is drawn.

If Jim draws a white marble in the first round, then we move down the tree to Erica's (or E's) first turn. Here she has a 1/4 chance of drawing the gold marble and a 3/4 chance of drawing a white marble.

The remaining possible turns are all represented likewise down the tree. Therefore, to answer the question, we need to determine the probability that Erica either draws the gold marble in round 2 or 4.

For Erica to draw the gold marble on turn two, first Jim has to draw white and then Erica has to draw gold. The probability of this occurring is

$$\frac{4}{5} \cdot \frac{1}{4} = \frac{1}{5}$$

For Erica to draw the gold marble on turn 4, we need Jim to draw white, then Erica to draw white, then Jim to draw white again, then Erica to draw gold. The probability of this occurring is

$$\frac{4}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{5}$$

Therefore, the total probability that Erica draws the gold marble is 0.2 + 0.2 = 0.4.

Solution 2 (Choosing Groups): Instead of imagining the two players drawing marbles one after another, an equivalent game would be to randomly assign Erica 2 marbles from the jar and to assign Jim the other 3 marbles from the jar.

There are 5 choose 2 (or 10) ways for Erica to be assigned two marbles from the jar. If we know that Erica is assigned the gold marble, then there are 4 choose 1 (or 4) ways she can be assigned one other marble from the jar. Therefore, we can conclude that the probability that Erica picks the gold marble is $\frac{4}{10}$, or 0.4.

Round 4: Perimeter, Area, and Volume

1. A rectangle and a square have the same perimeter. If the width of the rectangle is half the length of the square's side and the area of the square is 256 square units, what is the area of the rectangle?

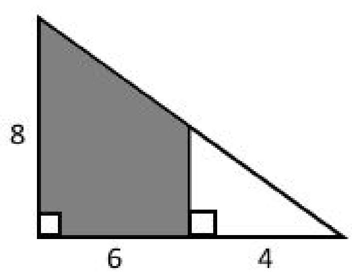
Solution: We know that the area of the square is 256 sq. units. Therefore the side of the square is 16 units. This implies that the width of the rectangle is 8 units.

Let x denote the length of the rectangle. Since the perimeter of the square is 64 units and since the rectangle and square have the same perimeter, we have that

$$2x + 2 \cdot 8 = 64$$
$$2x = 48$$
$$x = 24.$$

Hence, we have that the area of the rectangle is $24 \cdot 8 = 192$ square units.

2. Find the area of the shaded region in the following diagram.



Solution 1 (Trapezoid): Let x denote the height of the small white triangle in the diagram. Since the large triangle and the small white triangle are similar, we have that

$$\frac{4}{4+6} = \frac{x}{8}$$
$$32 = 10x$$
$$x = 3.2$$

We know that the area of a trapezoid with bases b_1 and b_2 and height h is given by

$$h \cdot \frac{b_1+b_2}{2}$$

Since the shaded area that we are interested in is a trapezoid, we have that its area is

$$6 \cdot \frac{8+3.2}{2} =$$
$$3 \cdot 11.2 = 33.6 \text{ sq units}$$

Solution 2 (Subtract Triangle): Begin exactly as in Solution 1 to determine that the value of x must be 3.2. Then note that the area of the large triangle is given by

$$\frac{1}{2} \cdot 8 \cdot 10 = 40 \text{ sq units}$$

We have that the area of the small white triangle is given by

$$\frac{1}{2} \cdot 4 \cdot 3.2 = 6.4 \text{ sq units}$$

Therefore, the area of the shaded region is equal to $40 - 6.4 = 33.6$ square units.

3. In a right rectangular prism, the length, width, and height are in the ratio 4:2:1. The volume of the prism is 512 cubic units. What is the surface area of the prism?

Solution: Let x denote the height of the prism. We have that the volume of the prism is 512 cubic units and we can express the volume in terms of x as

$$(x)(2x)(4x) = 512$$
$$8x^3 = 512$$
$$x^3 = 64$$
$$x = 4.$$

Therefore we have that the height is 4 units, the width is 8 units and the length is 16 units. We can now compute the surface area of the prism as

$$2[(4 \cdot 8) + (4 \cdot 16) + (8 \cdot 16)] =$$
$$2[32 + 64 + 128] =$$
$$2[224] = 448 \text{ square units.}$$

Team Round

1. On the real number line, let $A = \{x : \ -4 < 3 - 5x < 18\}$ and $B = \{x : \ |x-3| \ > \ 5\}$. If A^C denotes the complement of set A, graph the following region: $(A \cup B) \cap (A \cap B)^C$.

Solution: Begin by determining sets A and B. We have that set A contains all points satisfying

$$\begin{aligned} &-4 < 3 - 5x < 18 \\ &-7 < \ -5x < 15 \\ &1.2 \ > \ x \ > \ -3 \end{aligned}$$

Then we have that set B contains all points satisfying

$$\begin{aligned} &|x-3| \ > \ 5 \\ &x-3 \ > \ 5 \ \text{ or } \ 3-x \ > \ 5 \\ &x \ > \ 8 \ \text{ or } \ x \ < \ -2 \end{aligned}$$

Now recall that the set operation described in the question is precisely the symmetric difference of sets A and B. That is, it contains any point that is in either A or B, but not the points that belong to both A and B.

The only points that are contained in both A and B are $-3 < x < \ -2$. Therefore, the desired answer is $x \ < \ -3$ or $-2 < \ x \ < \ 1.2$ or $8 < \ x$.

2. Factor completely: $64x^6 - 1$

Solution: Begin by using the difference of squares formula:

$$64x^6 - 1 = (8x^3 + 1)(8x^3 - 1)$$

Next apply the sum of cubes formula and the difference of cubes formula:

$$\begin{aligned} &(8x^3 + 1)(8x^3 - 1) = \\ &(2x+1)(4x^2 - 2x + 1)(2x-1)(4x^2 + 2x + 1) \end{aligned}$$

3. A jar contains 5 quarters and 3 dimes. If Isaac shakes out two coins at random, what is the probability that the total value of the two coins is 35 cents?

Solution: The only way Isaac can get two coins that are not worth 35 cents is if he gets two quarters or two dimes. The probability he gets two quarters is given by

$$\frac{5}{8} \cdot \frac{4}{7} = \frac{5}{14} = \frac{10}{28}$$

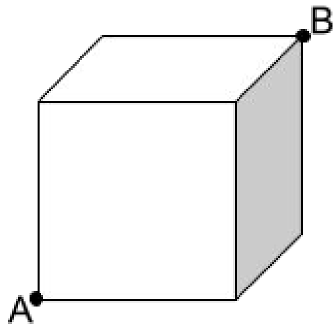
The probability that he gets two dimes is given by

$$\frac{3}{8} \cdot \frac{2}{7} = \frac{3}{28}$$

Therefore, the probability he gets 35 cents is given by

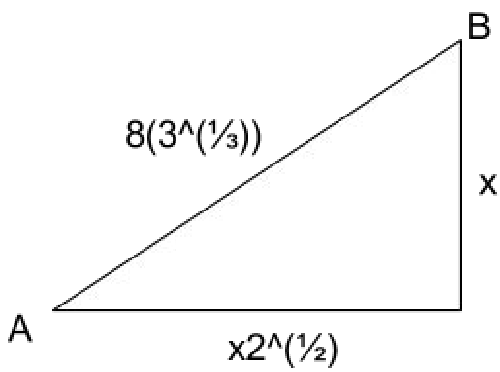
$$1 - \frac{10}{28} - \frac{3}{28} = \frac{15}{28}$$

4. In the following cube, if the straight line distance between A and B is $8\sqrt{3}$ units, what is the cube's surface area?



Solution: Let x denote the length of the side of the cube. Begin by examining the base of the cube. By the Pythagorean Theorem, we know that the diagonal of the base is given by $\sqrt{x^2+x^2} = x\sqrt{2}.$

This information combined with the given information allows us to draw the following diagram:



By again applying the Pythagorean Theorem, we see that

$$\left(x\sqrt{2}\right)^2 + x^2 = \left(8\sqrt{3}\right)^2$$
$$2x^2 + x^2 = 64 \cdot 3$$
$$3x^2 = 3 \cdot 64$$
$$x^2 = 64$$
$$x = 8$$

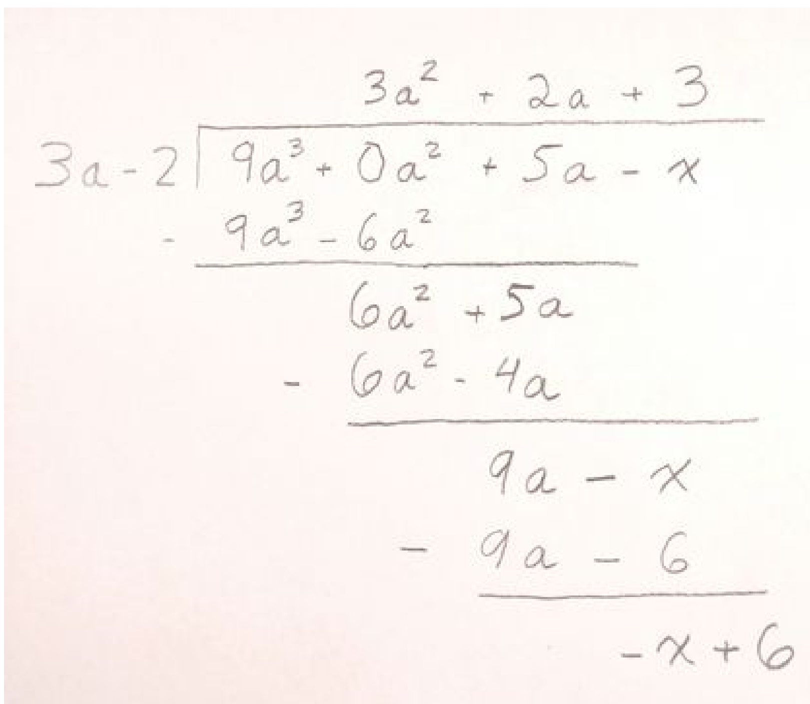
Therefore, the surface area of the cube is given by $6 \cdot 8^2 = 384$ square units.

5. How many integers strictly larger than 999 and strictly less than 10,000 do not contain the digit 4?

Solution: We are interested only in four digit integers, so we know that the thousands digit cannot be either 0 or 4. The hundreds, tens, and unit digit each can be any digit except for 4. That means in total there are $8 \cdot 9 \cdot 9 \cdot 9 = 5832$ possible four digit integers that do not contain the digit 4.

6. If $9a^3 + 5a - x$ is divided by $3a - 2$, the remainder is -2. Determine x.

Solution: We begin by performing the polynomial long division:



Since we know that the remainder is equal to -2, we have that

$$-x + 6 = -2$$
$$x = 8.$$

7. A sequence of 14 single digit numbers has the property that the sum of any three consecutive numbers is equal to 20. If the fourth number is 9 and the twelfth number is 7, what is the product of the first four digits?

Solution: We infer from the statement of the problem that the sequence of numbers must repeat after every three digits. We know this because if the sequence starts with w, x, y, z we have the following system of equations:

$$w + x + y = 20$$
$$x + y + z = 20$$

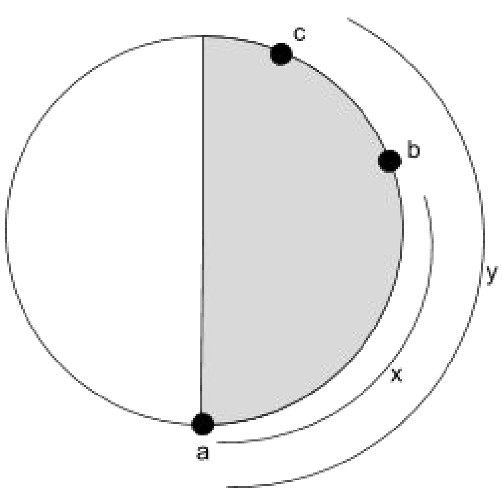
But if we subtract the second equation from the first we have that $w = z$.

Therefore, we can immediately conclude that the first number in the sequence is 9 since we are given that the fourth number is 9. We are also given that the twelfth number is 7, which means that the ninth number must be 7. By the same logic the sixth number must be 7, and likewise the third number must be 7.

Finally, since each set of three consecutive numbers adds to 20, we know that the second number must be 4, since 9 + 4 + 7 = 20. Therefore, the desired product is $9 \cdot 4 \cdot 7 \cdot 9 = 2268.$

8. Jack spins a fair spinner three times and draws a dot at the point of the circle where the tip of the spinner stops each time. After he is done spinning, what is the probability that he can find a semicircle on the circle which contains all three points?

Solution: Let a, b, and c denote the three points that Jack marks on the circle and let s denote the arc length of a semicircle. Begin by rotating the circle so that point a is at the circle's bottom. Next, measuring in a counterclockwise fashion, let x denote the arc length of arc ab and let y denote the arc length of arc ac (See figure below for example).



There are four ways that the three points can lie on the same semicircle.

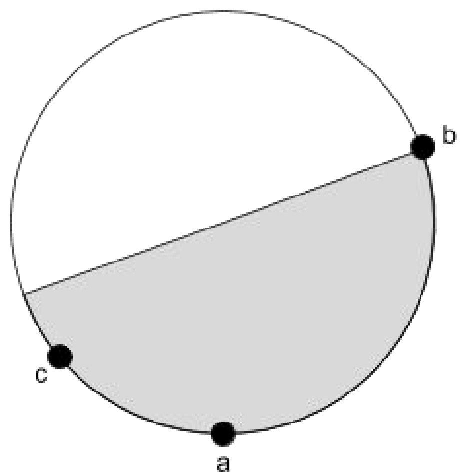
1. $x < s$ and $y < s$
2. $x < s$ and $y > (x + s)$
3. $x > s$ and $y > s$
4. $x > s$ and $y < (x - s)$

1. **x < s and y < s**

   In this scenario, we have a situation like the figure pictured on the previous page. Since both x and y are less than s, we know that b and c both are located on the right half of the circle.

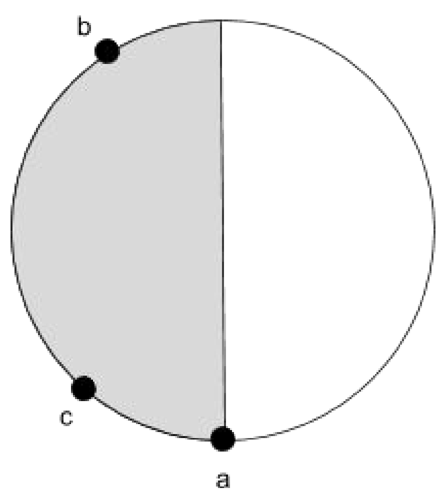
2. **x < s and y > (x + s)**

   In this scenario, point b will be located on the right half of the circle and point c will be on the left half, as seen below. Notice that the inequality on y guarantees that c will end up on the shaded semicircle.



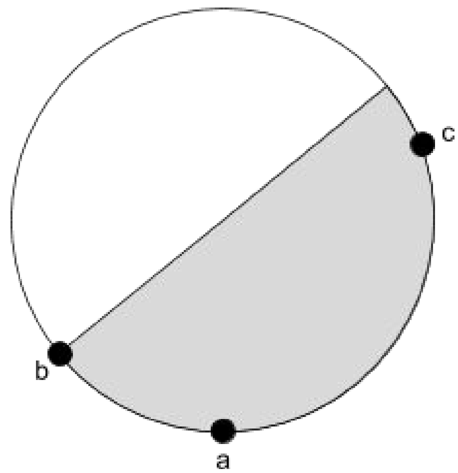
3. **x > s and y > s**

   In this scenario, both b and c both are located on the left half of the circle, as seen below:



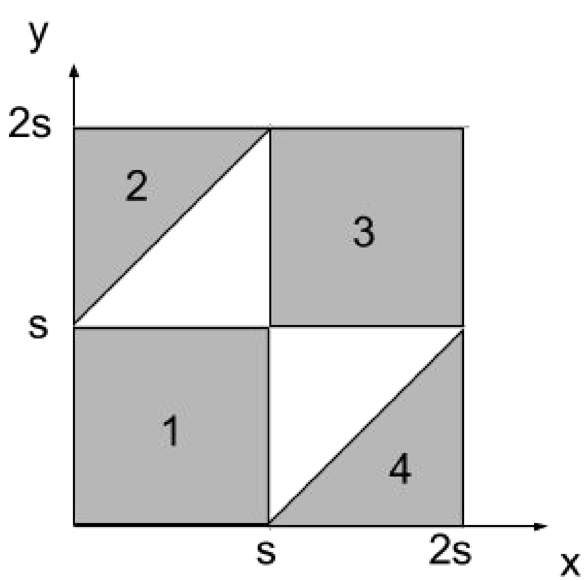
4. **x > s and y < (x - s)**

In this scenario, point b will be located on the left half of the circle and point c will be on the right half, as seen below. Notice that the inequality on y guarantees that c will end up on the shaded gray semicircle.



Now imagine the possible event space as a two dimensional graph, where the x-axis denotes the possible values for x and the y-axis denotes the possible values for y. Since s is the arc length of a semicircle, we know that x and y can each vary from 0 up to 2s. This event space is pictured below.

Each of the four situations examined above can now be shaded in on this graph. The four conditions again are:



1. $x < s$ and $y < s$
2. $x < s$ and $y > (x + s)$
3. $x > s$ and $y > s$
4. $x > s$ and $y < (x - s)$

Since precisely three quarters of the possible outcomes fall in the four shaded regions, we conclude that the probability that all three dots will lie on the same semicircle is 0.75.